

Error-Correcting Tournaments

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Abstract

Text of abstract We present a family of pairwise tournaments reducing k -class classification to binary classification. These reductions are provably robust against a constant fraction of binary errors, simultaneously matching the best possible computation $O(\log k)$ and regret $O(1)$.

The construction also works for robustly selecting the best of k -choices by tournament. We strengthen previous results by defeating a more powerful adversary than previously addressed while providing a new form of analysis. In this setting, the error correcting tournament has depth $O(\log k)$ while using $O(k \log k)$ comparators, both optimal up to a small constant.

Keywords: reductions, multiclass classification, cost-sensitive learning, tournaments, robust search

1. Introduction

We consider the classical problem of multiclass classification, where given an instance $x \in X$, the goal is to predict the most likely label $y \in \{1, \dots, k\}$, according to some unknown probability distribution.

A common general approach to multiclass learning is to reduce a multiclass problem to a set of binary classification problems [2, 7, 11, 12, 15].

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This approach is composable with any binary learning algorithm, including online algorithms, Bayesian algorithms, and even humans.

A key technique for analyzing reductions is *regret analysis*, which bounds the regret of the resulting multiclass classifier in terms of the average classification regret on the induced binary problems. Here *regret* (formally defined in Section 2) is the difference between the incurred loss and the smallest achievable loss on the problem, i.e., excess loss due to suboptimal prediction.

The most commonly applied reduction is one-against-all, which creates a binary classification problem for each of the k classes. The classifier for class i is trained to predict whether the label is i or not; predictions are then done by evaluating each binary classifier and randomizing over those that predict “yes,” or over all labels if all answers are “no”.

This simple reduction is *inconsistent*, in the sense that given optimal (zero-regret) binary classifiers, the reduction may not yield an optimal multiclass classifier in the presence of noise. Optimizing squared loss of the binary predictions instead of the 0/1 loss makes the approach consistent, but the resulting multiclass regret scales as $\sqrt{2kr}$ in the worst case, where r is the average squared loss regret on the induced problems. The Probing reduction [16] upper bounds r by the average binary classification regret. This composition gives a consistent reduction to binary classification, but it has a square root dependence on the binary regret (which is undesirable as regrets are between 0 and 1).

The probabilistic error-correcting output code approach (PECOC) [15] reduces k -class classification to learning $O(k)$ regressors on the interval $[0, 1]$, creating $O(k)$ binary examples per multiclass example at both training and test time, with a test time computation of $O(k^2)$. The resulting multiclass regret is bounded by $4\sqrt{r}$, removing the dependence on the number of classes k . When only a constant number of labels have non-zero probability given features, the computation can be reduced to $O(k \log k)$ per example [14].

This state of the problem raises several questions:

1. Is there a consistent reduction from multiclass to binary classification that does not have a square root dependence on r [18]? For example, an average binary regret of just 0.01 may imply a PECOC multiclass regret of 0.4.
2. Is there a consistent reduction that requires just $O(\log k)$ computation, matching the information theoretic lower bound?

The well-known $O(\log k)$ tree reduction distinguishes between the labels using a balanced binary tree, with each non-leaf node predicting

“Is the correct multiclass label to the left or not?” [10]. As shown in Section 3, this method is inconsistent.

3. Can the above be achieved with a reduction that only performs pairwise comparisons between classes?

One fear associated with the PECOC approach is that it creates binary problems of the form “What is the probability that the label is in a given random subset of labels?” which may be hard to solve. Although this fear is addressed by regret analysis (as the latter operates only on avoidable, excess loss), and is overstated in some cases [9, 14], it is still of some concern, especially with larger values of k .

The error-correcting tournament family presented here answers all of these questions in the affirmative. It provides an exponentially faster in k method for multiclass prediction with the resulting multiclass regret bounded by $5.5r$, where r is the average binary regret; and every binary classifier logically compares two distinct class labels.

The result is based on a basic observation that if a non-leaf node fails to predict its binary label, which may be unavoidable due to noise in the distribution, nodes between this node and the root should have no preference for class label prediction. Utilizing this observation, we construct a reduction, called the *filter tree*, which uses a $O(\log k)$ computation per multiclass example at both training and test time, and whose multiclass regret is bounded by $\log k$ times the average binary regret.

The decision process of a filter tree, viewed bottom up, can be viewed as a single-elimination tournament on a set of k players. Using multiple independent single-elimination tournaments is of no use as it does not affect the *average* regret of an adversary controlling the binary classifiers. Somewhat surprisingly, it is possible to have $\log k$ complete single-elimination tournaments between k players in $O(\log k)$ rounds, with no player playing twice in the same round. An *error-correcting tournament*, first pairs labels in such simultaneous single-elimination tournaments, followed by a final carefully weighted single-elimination tournament that decides among the $\log k$ winners of the first phase. As for the filter tree, test time evaluation can start at the root and proceed to a multiclass label with $O(\log k)$ computation.

This construction is also useful for the problem of robust search, yielding the first algorithm which allows the adversary to err a constant fraction of the time in the “full lie” setting [17], where a comparator can mis-sort any comparison. Previous work either applied to the “half lie” case where a comparator can fail to sort but can not actively mis-sort [6, 20], or to a “full lie” setting where an adversary has a fixed known bound on the

number of lies [17] or a fixed budget on the fraction of errors so far [5, 3]. Indeed, it might even appear impossible to have an algorithm robust to a constant fraction of full lie errors since an error can always be reserved for the last comparison. Repeating the last comparison $O(\log k)$ times defeats this strategy.

The result here is also useful for the actual problem of tournament construction in games with real players. Our analysis does not assume that errors are *i.i.d.* [8], or have known noise distributions [1] or known outcome distributions given player skills [13]. Consequently, the tournaments we construct are robust against severe bias such as a biased referee or some forms of bribery and collusion. Furthermore, the tournaments we construct are shallow, requiring fewer rounds than m -elimination bracket tournaments, which do not satisfy the guarantee provided here. In an *m-elimination bracket tournament*, bracket i is a single-elimination tournament on all players except the winners of brackets $1, \dots, i - 1$. After the bracket winners are determined, the player winning the last bracket m plays the winner of bracket $m - 1$ repeatedly until one player has suffered m losses (they start with $m - 1$ and $m - 2$ losses respectively). The winner moves on to pair against the winner of bracket $m - 2$, and the process continues until only one player remains. This method does not scale well to large m , as the final elimination phase takes $\sum_{i=1}^m i - 1 = O(m^2)$ rounds. Even for $k = 8$ and $m = 3$, our constructions have smaller maximum depth than bracketed 3-elimination. To see that the bracketed m -elimination tournament does not satisfy our goal, note that the second-best player could defeat the first player in the first single elimination tournament, and then once more in the final elimination phase to win, implying that an adversary need control only two matches.

Paper overview. We begin by defining the basic concepts and introducing some of the notation in Section 2. Section 3 shows that the simple divide-and-conquer tree approach is inconsistent, motivating the Filter Tree algorithm described in section 4 (which applies to more general cost-sensitive multiclass problems). Section 5 proves that the algorithm has the best possible computational dependence, and gives two upper bounds on the regret of the returned (cost-sensitive) multiclass classifier. Subsection 5.4 presents some experimental evidence that the Filter Tree is indeed a practical approach for multiclass classification.

Section 6 presents the error-correcting tournament family parametrized by an integer $m \geq 1$, which controls the tradeoff between maximizing robustness (m large) and minimizing depth (m small). Setting $m = 1$ gives the Filter Tree, while $m = 4 \ln k$ gives a (multiclass to binary) regret ratio

of 5.5 with $O(\log k)$ depth. Setting $m = ck$ gives regret ratio of $3 + O(1/c)$ with depth $O(k)$. The results here provide a nearly free generalization of earlier work [6] in the robust search setting, to a more powerful adversary that can missort as well as fail to sort.

Section 7 gives an algorithm independent lower bound of 2 on the regret ratio for large k . When the number of calls to a binary classifier is independent (or nearly independent) of the label predicted, we strengthen this lower bound to 3 for large k .

2. Preliminaries

Let D be the underlying distribution over $X \times Y$, where X is some observable feature space and $Y = \{1, \dots, k\}$ is the label space. The *error rate* of a classifier $f : X \rightarrow Y$ on D is given by

$$\text{err}(f, D) = \mathbf{Pr}_{(x,y) \sim D}[f(x) \neq y].$$

The *multiclass regret* of f on D is defined as

$$\text{reg}(f, D) = \text{err}(f, D) - \min_{g: X \rightarrow Y} \text{err}(g, D).$$

The algorithms here extend to the *cost-sensitive* case, where the underlying distribution D is over $X \times [0, 1]^k$. The *expected cost* of a classifier $f : X \rightarrow Y$ on D is

$$\ell(f, D) = \mathbf{E}_{(x,c) \sim D} [c_{f(x)}].$$

Here $c \in [0, 1]^k$ gives the cost of each of the k choices for x . As in the multiclass case, the *cost-sensitive regret* of f on D is defined as

$$\text{creg}(f, D) = \ell(f, D) - \min_{g: X \rightarrow Y} \ell(g, D).$$

3. Inconsistency of Divide and Conquer Trees

One standard approach for reducing multiclass learning to binary learning is to split the set of labels in half, learn a binary classifier to distinguish between the two subsets, and repeat recursively until each subset contains one label. Multiclass predictions are made by following a chain of classifications from the root down to the leaves.

This tree reduction transforms D into a distribution D_T over binary labeled examples by drawing a multiclass example (x, y) from D and a random non-leaf node i , and outputting instance $\langle x, i \rangle$ with label 1 if y is in the left

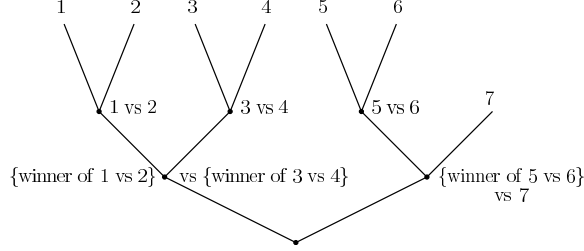


Figure 1: Filter Tree. Each node predicts whether the left or the right input label is more likely, conditioned on a given $x \in X$. The root node predicts the best label for x .

subtree of node i , and 0 otherwise. A binary classifier f for this induced problem gives a multiclass classifier $T(f)$, via a chain of binary predictions starting from the root.

The following theorem gives an example of a multiclass problem such that even if we have an optimal classifier for the induced binary problem at each node, the tree reduction does not yield an optimal multiclass predictor.

Theorem 1. *For all $k \geq 3$, for all binary trees over the labels, there exists a multiclass distribution D such that $\text{reg}(T(f^*), D) > 0$ for any $f^* = \arg \min_f \text{err}(f, D_T)$.*

Proof: Find a node with one subset corresponding to two labels and the other subset corresponding to a single label. (If the tree is perfectly balanced, simply let D assign probability 0 to one of the labels.) Since we can freely rename labels without changing the underlying problem, let the first two labels be 1 and 2, and the third label be 3.

Fix any $\epsilon \in (0, 1/12)$. Choose D with the property that labels 1 and 2 each have a $\frac{1}{4} + \epsilon$ chance of being drawn given x , and label 3 is drawn with the remaining probability of $\frac{1}{2} - 2\epsilon$. Under this distribution, the fraction of examples for which label 1 or 2 is correct is $\frac{1}{2} + 2\epsilon$, so any minimum error rate binary predictor must choose either label 1 or label 2. Each of these choices has an error rate of $\frac{3}{4} - \epsilon$. The optimal multiclass predictor chooses label 3 and suffers an error rate of $\frac{1}{2} + 2\epsilon$, implying that the regret of the tree classifier based on the optimal binary classifier is $\frac{1}{4} - 3\epsilon$, which is strictly greater than 0 as $\epsilon < 1/12$. ■

4. The Filter Tree Algorithm

The Filter Tree algorithm, illustrated by Figure 1, is equivalent to a single-elimination tournament on the set of labels, structured as a binary

Algorithm 1 Filter tree training (multiclass training set S , binary learning algorithm **Learn**)

Define $y_u = 1$ if label y is in the left subtree of node u ; otherwise $y_u = 0$.

for each non-leaf node n in order from leaves to root **do**

 Set $S_n = \emptyset$

for each $(x, y) \in S$ such that $y \in L(T_n)$ and all nodes u on the path $n \rightsquigarrow y$ predict y_u given x **do**

 add (x, y_n) to S_n

end for

 Let $f_n = \text{Learn}(S_n)$

end for

return $f = \{f_n\}$

tree T over the labels. In the first round, the labels are paired according to the lowest level of the tree, and a classifier is trained for each pair to predict which of the two labels is more likely. (The labels that don't have a pair in a given round, win that round for free.) The winning labels from the first round are in turn paired in the second round, and a classifier is trained to predict whether the winner of one pair is more likely than the winner of the other. The process of training classifiers to predict the best of a pair of winners from the previous round is repeated until the root classifier is trained.

The key trick in the training stage (Algorithm 1) is to form the right training set at each interior node. We use T_n to denote the subtree of T rooted at node n , and $L(T)$ to denote the set of leaves in the tree T . A training example for node n is formed conditioned on the predictions of classifiers in the round before it. Thus the learned classifiers from the first level of the tree are used to “filter” the distribution over examples reaching the second level of the tree.

Given x and classifiers at each node, every edge in T is identified with a unique label. The optimal decision at any non-leaf node is to choose the input edge (label) that is more likely according to the true conditional probability. This can be done by using the outputs of classifiers in the round before it as a filter during the training process: For each observation, we set the label to 0 if the left parent's output matches the multiclass label, 1 if the right parent's output matches, and reject the example otherwise.

Algorithm 2 extends this idea to the cost-sensitive multiclass case where each choice has a different associated cost, as defined in Section 2. The algo-

Algorithm 2 Cost-sensitive filter tree training (cost-sensitive training set S , importance weighted binary learner **Learn**)

```

1: for each non-leaf node  $n$  in the order from leaves to root do
2:   Set  $S_n = \emptyset$ 
3:   for each example  $(x, c_1, \dots, c_k) \in S$  do
4:     Let  $a$  and  $b$  be the two classes input to  $n$ 
5:      $S_n \leftarrow S_n \cup \{(x, \arg \min\{c_a, c_b\}, \underbrace{|c_a - c_b|}_{w_n(x,c)})\}$ 
6:   end for
7:   Let  $f_n = \text{Learn}(S_n)$ 
8: end for
9: return  $f = \{f_n\}$ 

```

algorithm relies upon an *importance weighted* binary learning algorithm **Learn**, which takes examples of the form (x, y, w) , where $x \in X$ is a feature vector used for prediction, $y \in \{0, 1\}$ is a binary label, and $w \in [0, \infty)$ is the importance any classifier pays if it doesn't predict y on x . The importance weighted problem can be further reduced to binary classification using the Costing reduction [21], which alters the underlying distribution using rejection sampling on the importances. This is the reduction we use here.

The testing algorithm is the same for both multiclass and cost-sensitive variants, and is very simple: Given a test example $x \in X$, we output the label y such that every classifier on the path from the root to y prefers y .

5. Filter Tree Analysis

Before analyzing the regret of the algorithm, we note its computational characteristics.

5.1. Computational Complexity

Since the algorithm is a reduction, we count the computational complexity of the reduction itself, assuming that oracle calls take unit time.

Algorithm 1 requires $O(\log k)$ computation per multiclass example, by searching for the correct leaf in $O(\log k)$ time, then filtering back toward the root. This matches the information theoretic lower bound since simply reading one of k labels requires $\lceil \log_2 k \rceil$ bits.

Algorithm 2 requires $O(k)$ computation per cost-sensitive example, because there are $k - 1$ nodes, each requiring constant computation per example. Since any method must read the k costs, this bound is tight.

Testing requires $O(\log k)$ computation per example to descend a binary tree. Any method must write out $\lceil \log_2 k \rceil$ bits to specify its prediction.

5.2. Regret Analysis

Algorithm 2 transforms each cost-sensitive multiclass example (line 3) into importance weighted binary labeled examples (line 5), one for every non-leaf node n in the tree. This process implicitly transforms the underlying distribution D over cost-sensitive multiclass examples into a distribution D_n over importance weighted binary examples at each n .

We can further reduce from importance weighted binary classification to binary classification using the Costing reduction [21], which alters each D_n using rejection sampling on the importance weights. This composition further transforms D_n into a distribution D'_n over binary examples.

Let f_n be a classifier for the binary classification problem induced at node n . The relevant quantity is the *average binary regret*,

$$\text{reg}(f, D') = \frac{1}{\sum_{n \in T} W_n} \sum_{n \in T} \text{reg}(f_n, D'_n) W_n, \quad (1)$$

where $W_n = \mathbf{E}_{(x,c) \sim D} w_n(x, c)$, and $w_n(x, c)$ is the importance weight formed in line 5 of Algorithm 2 (the difference in cost between the two labels that node n chooses between on x). This quantity, which is just the average *importance weighted* binary regret of f_n on D_n , is induced by the reduction (Algorithm 2).

The core theorem below relates $\text{reg}(f, D')$ to the regret of the resulting cost-sensitive classifier $T(f)$ on D . Again, given a test example $x \in X$, the classifier $T(f)$ returns the unique label y such that every f_n on the path from the root to y prefers y .

This type of analysis is similar to Boosting: At each round n , the booster creates an input distribution D_n and calls a weak learning algorithm to obtain a classifier f_n , which has some error rate on D_n . The distribution D_n depends on the classifiers returned by the oracle in previous rounds. The accuracy of the final classifier on the original distribution D is analyzed in terms of these error rates.

Theorem 2. *For all binary classifiers f and all cost-sensitive multiclass distributions D ,*

$$\text{creg}(T(f), D) \leq \text{reg}(f, D') \sum_{n \in T} W_n,$$

where $W_n = \mathbf{E}_{(x,c) \sim D} w_n(x, c)$, and $w_n(x, c)$ is the importance weight formed in line 5 of Algorithm 2 (the difference in cost between the two labels that node n chooses between on x).

Before proving the theorem, we state the corollary for multiclass classification.

Corollary 3. *For all binary classifiers f and multiclass distributions D ,*

$$\text{reg}(T(f), D) \leq d \text{reg}(f, D'),$$

where d is the depth of the tree T .

Since all importance weights are either 0 or 1, we don't need to apply Costing in the multiclass case. The proof of the corollary given the theorem is simple since for any (x, y) , the induced (x, c) has at most one node per level with induced importance weight 1; all other importance weights are 0. Therefore, $\sum_n w_n(x, c) \leq d$.

Theorem 4 provides an alternative bound for cost-sensitive classification. It is the first known bound giving a worst-case dependence of less than k .

Theorem 4. *For all binary classifiers f and all cost-sensitive k -class distributions D ,*

$$\text{creg}(T(f), D) \leq k \text{reg}(f, D')/2,$$

where $T(f)$ and D' are as defined above.

A simple example in Section 5.3 shows that this bound is essentially tight.

The proof of Theorem 2 uses the following folk theorem from [21].

Theorem 5. (Translation Theorem [21]) *For any importance-weighted distribution P , there exists a constant $\langle c \rangle = \mathbf{E}_{(x,y,c) \sim P}[c]$ such that for any classifier f ,*

$$\mathbf{E}_{(x,y,c) \sim P}[c \cdot \mathbf{1}(f(x) \neq y)] = \langle c \rangle \mathbf{E}_{(x,y,c) \sim P'}[\mathbf{1}(f(x) \neq y)],$$

where $P'(x, y, c) = \frac{c}{\langle c \rangle} P(x, y, c)$.

Thus choosing f to minimize the error rate under P' is equivalent to choosing f to minimize the expected cost under P . The Costing [21] reduction uses rejection sampling according to the weights to draw examples from P' given examples drawn from P .

The remainder of this section proves Theorems 2 and 4.

Proof of Theorem 2: It is sufficient to prove the claim for any $x \in X$ because that implies that the result holds for all expectations over x .

Conditioned on the value of x , each label y has a distribution over costs with an expected value of $\mathbf{E}_{c \sim D|x}[c_y]$. The zero regret cost-sensitive classifier predicts according to $\arg \min_y \mathbf{E}_{c \sim D|x}[c_y]$. Suppose that $T(f)$ predicts y' on x , inducing cost-sensitive regret

$$\text{creg}(y', D | x) = \mathbf{E}_{c \sim D|x}[c_{y'}] - \min_y \mathbf{E}_{c \sim D|x}[c_y].$$

First, we show that the sum over the binary problems of the importance weighted regret is at least $\text{creg}(y', D | x)$, using induction starting at the leaves. The induction hypothesis is that the sum of the regrets of importance-weighted binary classifiers in any subtree bounds the regret of the subtree output.

For node n , each importance weighted binary decision between class a and class b has an importance weighted regret which is either 0 or $r_n = |\mathbf{E}_{\tilde{c} \sim D|x}[c_a - c_b]| = |\mathbf{E}_{\tilde{c} \sim D|x}[c_a] - \mathbf{E}_{\tilde{c} \sim D|x}[c_b]|$, depending on whether the prediction is correct or not.

Assume without loss of generality that the predictor outputs class b . The regret of the subtree T_n rooted at n is given by

$$r_{T_n} = \mathbf{E}_{\tilde{c} \sim D|x}[c_b] - \min_{y \in L(T_n)} \mathbf{E}_{\tilde{c} \sim D|x}[c_y].$$

As a base case, the inductive hypothesis is trivially satisfied for trees with one label. Inductively, assume that $\sum_{n' \in L} r_{n'} \geq r_L$ and $\sum_{n' \in R} r_{n'} \geq r_R$ for the left subtree L of n (providing a) and the right subtree R (providing b).

There are two possibilities. Either the minimizer comes from the leaves of L or the leaves of R . The second possibility is easy since we have

$$r_{T_n} = \mathbf{E}_{\tilde{c} \sim D|x}[c_b] - \min_{y \in L(R)} \mathbf{E}_{\tilde{c} \sim D|x}[c_y] = r_R \leq \sum_{n' \in R} r_{n'} \leq \sum_{n' \in T_n} r_{n'},$$

proving the induction.

For the first possibility,

$$\begin{aligned} r_{T_n} &= \mathbf{E}_{\tilde{c} \sim D|x}[c_b] - \min_{y \in L(L)} \mathbf{E}_{\tilde{c} \sim D|x}[c_y] \\ &= \mathbf{E}_{\tilde{c} \sim D|x}[c_b] - \mathbf{E}_{\tilde{c} \sim D|x}[c_a] + \mathbf{E}_{\tilde{c} \sim D|x}[c_a] - \min_{y \in L(L)} \mathbf{E}_{\tilde{c} \sim D|x}[c_y] \\ &= \mathbf{E}_{\tilde{c} \sim D|x}[c_b] - \mathbf{E}_{\tilde{c} \sim D|x}[c_a] + r_L \\ &\leq r_n + \sum_{n' \in L} r_{n'} \leq \sum_{n' \in T_n} r_{n'}, \end{aligned}$$

which completes the induction. The inductive hypothesis for the root is that $\text{creg}(y', D|x) \leq \sum_{n \in T} r_n$.

Using the folk theorem from [21] (Theorem 5 in this paper), each r_n is bounded by

$$r_n \leq W_n \text{reg}(f_n, D'_n).$$

Plugging this in and using Definition (1), we get the theorem. ■

The proof of Theorem 4 makes use of the following lemma. Consider a filter tree T on k labels, evaluated on a cost-sensitive multiclass example with cost vector $c \in [0, 1]^k$. Let S_T be the sum of importances over all nodes in T , and I_T be the sum of importances over the nodes where the class with the larger cost was selected for the next round. Let c_T denote the cost of the winner chosen by T .

Lemma 6. *For any $c \in [0, 1]^k$, $S_T + c_T \leq I_T + \frac{k}{2}$.*

Proof: The inequality follows by induction, the result being immediate when $k = 2$. Assume that the claim holds for the two subtrees, L and R , providing their respective inputs l and r to the root of T , and T outputs r without loss of generality. Using the inductive hypotheses for L and R , we get $S_T + c_T = S_L + S_R + |c_r - c_l| + c_r \leq I_L + I_R + \frac{k}{2} - c_l + |c_r - c_l|$.

If $c_r \geq c_l$, we have $I_T = I_L + I_R + (c_r - c_l)$, and

$$S_T + c_T \leq I_T + \frac{k}{2} - c_l \leq I_T + \frac{k}{2},$$

as desired. If $c_r < c_l$, we have $I_T = I_L + I_R$ and $S_T + c_T \leq I_T + \frac{k}{2} - c_r \leq I_T + \frac{k}{2}$, completing the proof. ■

Proof of Theorem 4: Fix $(x, c) \in X \times [0, 1]^k$ and take the expectation over the draw of (x, c) from D as the last step.

Consider a filter tree T evaluated on (x, c) using a given binary classifier f . As before, let S_T be the sum of importances over all nodes in T , and I_T be the sum of importances over the nodes where f made a mistake. Recall that the regret of T on (x, c) , denoted in the proof by reg_T , is the difference between the cost of the tree's output and the smallest cost c^* . The importance-weighted binary regret of f on (x, c) is simply I_T/S_T . Since the expected importance is upper bounded by 1, I_T/S_T also bounds the binary regret of f .

The inequality we need to prove is $\text{reg}_T S_T \leq \frac{k}{2} I_T$. The proof is by induction on k , the result being trivial if $k = 2$. Assume that the assertion

holds for the two subtrees, L and R , providing their respective inputs l and r to the root of T . (The number of classes in L and R can be taken to be even, by splitting the odd class into two classes with the same cost as the split class, which has no effect on the quantities in the theorem statement.)

Let the best cost c^* be in the left subtree L . Suppose first (**Case 1**) that T chooses r and $c_r > c_l$. Let $w = c_r - c_l$. We have $\text{reg}_L = c_l - c^*$ and $\text{reg}_T = c_r - c^* = \text{reg}_L + w$. The left hand side of the inequality is thus

$$\begin{aligned}
\text{reg}_T S_T &= (\text{reg}_L + w)(S_R + S_L + w) \\
&= w(\text{reg}_L + S_R + S_L + w) + \text{reg}_L(S_L + S_R) \\
&\leq w(\text{reg}_L + I_R + I_L - c_r - c_l + w + \frac{k}{2}) + \text{reg}_L(I_R + I_L - c_l - c_r + \frac{k}{2}) \\
&\leq \frac{k}{2}w + I_R(w + \text{reg}_L) + I_L(w + \text{reg}_L) + \text{reg}_L\left(\frac{k}{2} - c_r - c_l\right) \\
&\leq \frac{k}{2}w + I_R(w + \text{reg}_L) + I_L\left(w + \text{reg}_L + \frac{k}{2} - c_r - c_l\right) \\
&\leq \frac{k}{2}w + I_R(w + \text{reg}_L) + \frac{k}{2}I_L \leq \frac{k}{2}(w + I_R + I_L) = \frac{k}{2}I_T.
\end{aligned}$$

The first inequality follows from lemma 6. The second and fourth follow from $w(\text{reg}_L - c_l - c_r + w) \leq 0$. The third follows from $\text{reg}_L \leq I_L$. The last follows from $\text{reg}_T \leq \frac{k}{2}$ for $k \geq 2$.

The proofs for the remaining three cases ($c_T = c_l < c_r$, $c_T = c_l > c_r$, and $c_l > c_r = c_T$) use the same machinery as the proof above.

Case 2: T outputs l , and $c_l < c_r$. In this case $\text{reg}_T = \text{reg}_L = c_l - c^*$. The left hand side can be rewritten as

$$\begin{aligned}
\text{reg}_T S_T &= \text{reg}_L(S_R + S_L + c_r - c_l) = \text{reg}_L S_L + \text{reg}_L(S_R + c_r - c_l) \\
&\leq \text{reg}_L\left(I_L + I_R - 2c_l + \frac{k}{2}\right) \leq I_R + \text{reg}_L\left(I_L - 2c_l + \frac{k}{2}\right) \\
&\leq I_R + I_L\left(\text{reg}_L - 2c_l + \frac{k}{2}\right) \leq I_R + \frac{k}{2}I_L \leq \frac{k}{2}I_T.
\end{aligned}$$

The first inequality follows from the lemma, the second from $\text{reg}_L \leq 1$, the third from $\text{reg}_L \leq I_L$, the fourth from $-c_L - c^* < 0$, and the fifth because $I_T = I_L + I_R$.

Case 3: T outputs l , and $c_l > c_r$. We have $\text{reg}_T = \text{reg}_L = c_l - c^*$. The left

hand side can be written as

$$\begin{aligned}
\text{reg}_T S_T &= \text{reg}_L(S_R + S_L + c_l - c_r) \\
&\leq \frac{|L|}{2} I_L + \text{reg}_L \left(I_R + \frac{k - |L|}{2} - c_r + c_l - c_r \right) \\
&\leq \frac{k}{2} I_L + I_R + (c_l - 2c_r) \leq \frac{k}{2} (I_L + I_R + (c_l - c_r)) = \frac{k}{2} I_T,
\end{aligned}$$

The first inequality follows from the inductive hypothesis and the lemma, the second from $\text{reg}_L < 1$ and $\text{reg}_L < I_L$, and the third from $c_r > 0$ and $k/2 > 1$.

Case 4: T outputs r , and $c_l > c_r$. Let $w = c_l - c_r$. We have $\text{reg}_T = c_r - c^* = \text{reg}_L - w$. The left hand side can be written as

$$\begin{aligned}
\text{reg}_T S_T &= (\text{reg}_L - w)(S_R + S_L + w) \\
&= \text{reg}_L S_L - w S_L + (\text{reg}_L - w)(S_R + w) \\
&\leq \frac{|L|}{2} I_L - w \left(I_L + \frac{|L|}{2} - c_l \right) + (\text{reg}_L - w) \left(I_R + c_l - 2c_r + \frac{k - |L|}{2} \right) \\
&\leq \frac{|L|}{2} I_L - w \left(I_L + \frac{|L|}{2} - c_l \right) + (I_L - w) \frac{k - |L|}{2} \\
&\quad + (\text{reg}_L - w)(I_R + c_l - 2c_r) \\
&\leq \frac{k}{2} (I_L + I_R) - w \frac{k}{2} - w(I_L - c_l) + (\text{reg}_L - w)(c_l - 2c_r).
\end{aligned}$$

The first inequality follows from the inductive hypothesis and the lemma, the second from $\text{reg}_L \leq I_L$, and the third from $\text{reg}_L \leq \frac{k}{2}$.

The last three terms are upper bounded by $-w - w\text{reg}_L + wc_l + \text{reg}_L c_l - 2c_r \text{reg}_L - wc_l + 2wc_r \leq -w - \text{reg}_L(c_r + c_l) + \text{reg}_L c_l + 2wc_r \leq -w - (c_l - c^*)c_r + wc_r + (c_l - c_r)c_r \leq 0$, and thus can be ignored, yielding $\text{reg}_T S_T \leq \frac{k}{2} (I_L + I_R) = \frac{k}{2} I_T$, which completes the proof. Taking the expectation over (x, c) completes the proof. ■

5.3. Tightness of Theorem 4

The following simple example shows that the theorem is essentially tight. Let k be a power of two, and let every label have cost 0 if it is even, and 1 otherwise. The tree structure is a complete binary tree of depth $\log k$ with the nodes being paired in the order of their labels. Suppose that all pairwise classifications are correct, except that class k wins all its $\log k$ games leading to cost-sensitive multiclass regret 1. We have $\text{reg}_T = 1$, $S_T = \frac{k}{2} + \log k - 1$, and $I_T = \log k$, leading to the regret ratio $\text{reg}_T S_T / I_T = \Omega(\frac{k}{2 \log k})$, almost matching the theorem's bound of $\frac{k}{2}$ on the ratio.

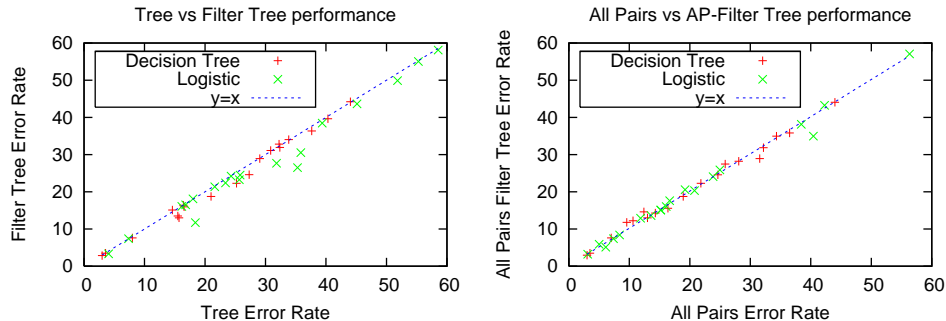


Figure 2: Error rates (in %) of **Tree** versus **Filter-Tree** (top) and All-Pairs versus All-Pairs Filter Tree (top) on several different datasets with a decision tree or logistic regression classifier.

5.4. Experimental Results

There is a variant of the Filter Tree algorithm, which has a significant difference in performance in practice. Every classification at any node n is essentially between two labels computed at test time, implying that we could simply learn one classifier for every pair of labels that could reach n at test time. (Note that a given pair of labels can be compared only at a single node, namely their least common ancestor in the tree.) The conditioning process and the tree structure gives us a better analysis than is achievable with the All-Pairs approach [12]. This variant uses more computation and requires more data but often maximizes performance when the form of the classifier is constrained.

We compared the performance of Filter Tree and its All-Pairs variant described above to the performance of All-Pairs and the Tree reduction, on a number of publicly available multiclass datasets [4]. Some datasets came with a standard training/test split: **isolet** (isolated letter speech recognition), **optdigits** (optical handwritten digit recognition), **pendigits** (pen-based handwritten digit recognition), **satimage**, and **soybean**. For all other datasets, we reported the average result over 10 random splits, with 2/3 of the dataset used for training and 1/3 for testing. (The splits were the same for all methods.)

If computation is constrained and we can afford only $O(\log k)$ computation per multiclass prediction, the Filter Tree dominates the Tree reduction, as shown in Figure 2.

If computation is relatively unconstrained, All-Pairs and the All-Pairs Filter Tree are reasonable choices. The comparison in Figure 2 shows that there the All-Pairs Filter Tree yields similar prediction performance while

using only $O(k)$ computation instead of $O(k^2)$.

Test error rates using decision trees (J48) and logistic regression as binary classifier learners are reported in Table A.1, using Weka’s implementation with default parameters [19]. The lowest error rate in each row is shown in bold, although in some cases the difference is insignificant.

6. Error-Correcting Tournaments

In this section, we extend filter trees to m -elimination tournaments, also called $(m - 1)$ -error-correcting tournaments. As this section builds on Sections 4 and 5, understanding them is required before reading this section. For simplicity, we work with only the multiclass case. An extension for cost-sensitive multiclass problems is possible using the importance weighting techniques of the previous section.

6.1. Algorithm Description

An m -elimination tournament operates in two phases.

The first phase consists of m single-elimination tournaments over the k labels where a label is paired against another label at most once per round. Consequently, only one of these single elimination tournaments has a simple binary tree structure; see, for example, Figure 3 for an $m = 3$ elimination tournament on $k = 8$ labels. There is substantial freedom in how the pairings of the first phase are done; our bounds depend on the depth of any mechanism which pairs labels in m distinct single elimination tournaments. One such explicit mechanism is given in [6]. Note that once an example has lost m times, it is eliminated and no longer influences training at the nodes closer to the root.

The second phase is a final elimination phase, where we select the winner from the m winners of the first phase. It consists of a redundant single-elimination tournament, where the degree of redundancy increases as the root is approached. To quantify the redundancy, let every subtree Q have a *charge* c_Q equal to the number of leaves under the subtree. First phase winners at the leaves of the final elimination tournament have charge 1. For any non-leaf node comparing the outputs of subtrees A and B , the importance weight of a binary example created at the node is set to either c_A or c_B , depending on whether the label comes from B or A . In tournament applications, an importance weight can be expressed by playing games repeatedly where the winner of A must beat the winner of B c_B times to advance, and vice versa. When the two labels compared are the same, the importance

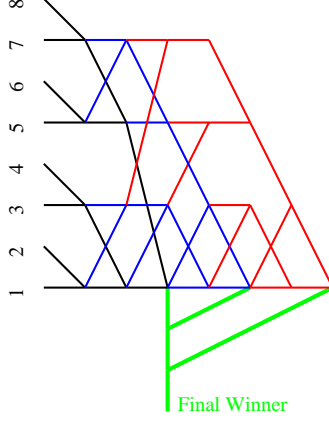


Figure 3: An example of a 3-elimination tournament on $k = 8$ players. There are $m = 3$ distinct single elimination tournaments in first phase—one in black, one in blue, and one in red. After that, a final elimination phase occurs over the three winners of the first phase. The final elimination tournament has an extra weighting on the nodes, detailed in the text.

weight is set to 0, indicating there is no preference in the pairing amongst the two choices.

6.2. Error Correcting Tournament Analysis

A key concept throughout this section is the *importance depth*, defined as the worst-case length (number of games) of the overall tournament, where importance-weighted matches in the final elimination phase are played as repeated games. In Theorem 11 we prove a bound on the importance depth.

The computational bound per example is essentially just the importance depth.

Theorem 7. (Structural Depth Bound) *For any m -elimination tournament, the training and test computation is $O(m + \ln k)$ per example.*

Proof: The proof is by simplification of the importance depth bound (theorem 11), which bounds the sum of importance weights at all nodes in the tournament.

To see that the importance depth controls the computation, first note that the importance depth bounds the tournament depth since all importance weights are at least 1. At training time, any one example is used at most once per tournament level starting at the leaves. At testing time, an

unlabeled example can have its label determined by traversing the structure from root to leaf. ■

6.3. Regret analysis

Our regret theorem is the analogue of Corollary 3 for error-correcting tournaments, and the notation is as defined there. As in the previous section, the reduction transforms a multiclass distribution D into an induced distribution D' over binary labeled examples. As before, $T(f)$ denotes the multiclass classifier induced by a given binary classifier f and tournament structure T .

It is useful to have the notation $\lceil m \rceil_2$ for the smallest power of 2 larger than or equal to m .

Theorem 8. (Main Theorem) *For all distributions D over k -class examples, all binary classifiers f , all m -elimination tournaments T , the ratio of $\text{reg}(T(f), D)$ to $\text{reg}(f, D')$ is upper bounded by*

$$\begin{cases} 2 + \frac{\lceil m \rceil_2}{m} + \frac{k}{2m} & \text{for all } m \geq 2 \text{ and } k > 2 \\ 4 + \frac{2 \ln k}{m} + 2\sqrt{\frac{\ln k}{m}} & \text{for all } k \leq 2^{62} \text{ and } m \leq 4 \log_2 k \end{cases}$$

The first case shows that a regret ratio of 3 is achievable for very large m . The second case is the best bound for cases of common interest. For $m = 4 \ln k$ it gives a ratio of 5.5.

Proof: The proof holds for each input x , and hence in expectation over x .

Fix x , and let $p_y = D(y \mid x)$ for $y \in \{1, \dots, k\}$. We can define the regret of any label y as $r_y = p^* - p_y$, where $p^* = \max_{a \in \{1, \dots, k\}} p_a$.

The regret of a node n comparing labels a and b from subtrees A and B , and outputting a , is

$$r_n = c_B(p_b - p_a)_+,$$

where we use the predicate $(z)_+ = \max(z, 0)$. Thus r_n is 0 if n outputs the more likely label. If n is in a first phase tournament, $r_n = (p_b - p_a)_+$.

Finally, the regret of a subtree T is defined as $r_T = \sum_{n \in T} r_n$.

The first part of the proof is by induction on the tree structure F of the final phase. The invariant for a subtree Q of F won by label a is

$$c_Q r_a \leq r_Q + \sum_{w \in L(Q)} r_w,$$

where w is the winner of a first phase single-elimination tournament W .

When Q is a leaf w of F , we have $c_Q r_w = r_w \leq r_W$, where the inequality is from Corollary 3 noting that the depth of W times the average regret over the nodes in W is r_W .

Assume inductively that the hypothesis holds at node n comparing labels a and b from subtrees A and B , and outputting a : $c_A r_a \leq r_A + \sum_{w \in L(A)} r_W$ and $c_B r_b \leq r_B + \sum_{w \in L(B)} r_W$. We have $r_Q + \sum_{w \in L(Q)} r_W \geq r_n + c_A r_a + c_B r_b$ by the inductive hypothesis.

Now, there are two cases: Either $p_b \leq p_a$, in which case $r_n = 0$ and $c_A r_a + c_B r_b \geq c_A r_a + c_B r_a \geq C_Q r_a$, as desired. Or $p_b > p_a$, in which case $r_n = c_B(p_b - p_a)$ and thus

$$\begin{aligned} r_n + c_A r_a + c_B r_b &= c_B p_b - c_B p_a + c_A p^* - c_A p_a + c_B p^* - c_B p_b \\ &= p^* c_Q - p_a C_Q = (p^* - p_a) C_Q = r_a C_Q, \end{aligned}$$

finishing the induction.

Finally, letting y be the prediction of $T(f)$ on x ,

$$m \operatorname{reg}(T(f), D \mid x) = c_F r_y \leq r_F + \sum_{w \in L(F)} r_W \leq d \operatorname{reg}(f, D' \mid x),$$

where d is the maximum importance depth. Applying the importance depth theorem (Theorem 11) and algebra completes the proof. ■

The depth bound follows from the following three lemmas.

Lemma 9. (First Phase Depth bound) *The importance depth of the first phase tournament is bounded by the minimum of*

$$\begin{cases} \lceil \log_2 k \rceil + m \lceil \log_2 (\lceil \log_2 k \rceil + 1) \rceil \\ 1.5 \lceil \log_2 k \rceil + 3m + 1 \\ \lceil \frac{k}{2} \rceil + 2m \\ \text{For } k \leq 2^{62} \text{ and } m \leq 4 \log_2 k, 2(m - 1) + \ln k + \sqrt{\ln k} \sqrt{\ln k + 4(m - 1)}. \end{cases}$$

Proof: The depth of the first phase is bounded by the classical problem of robust minimum finding with low depth. The first three cases hold because any such construction upper bounds the depth of an error-correcting tournament, and one such construction has these bounds [6].

For the fourth case, we construct the depth bound by analyzing a continuous relaxation of the problem. The relaxation allows the number of labels remaining in each single elimination tournament of the first phase to be broken into fractions. Relative to this version, the actual problem has two important discretizations:

1. When a single-elimination tournament has only a single label remaining, it enters the next single elimination tournament. This can have the effect of *decreasing* the depth compared to the continuous relaxation.
2. When a single-elimination tournament has an odd number of labels remaining, the odd label does not play that round. Thus the number of players does not quite halve, potentially *increasing* the depth compared to the continuous relaxation.

In the continuous version, tournament i on round d has $\frac{\binom{d}{i-1}k}{2^d}$ labels, where the first tournament corresponds to $i = 1$. Consequently, the number of labels remaining in any of the tournaments is $\frac{k}{2^d} \sum_{i=1}^m \binom{d}{i-1}$. We can get an estimate of the depth by finding the value of d such that this number is 1.

This value of d can be found using the Chernoff bound. The probability that a coin with bias $1/2$ has $m-1$ or fewer heads in d coin flips is bounded by $m^{-2d(\frac{1}{2} - \frac{m-1}{d})^2}$, and the probability that this occurs in k attempts is bounded by k times that. Setting this value to 1, we get $\ln k = 2d(\frac{1}{2} - \frac{m-1}{d})^2$. Solving the equation for d , gives $d = 2(m-1) + \ln k + \sqrt{4(m-1)\ln k + (\ln k)^2}$. This last formula was verified computationally for $k < 2^{62}$ and $m < 4\log_2 k$ by discretizing k into factors of 2 and running a simple program to keep track of the number of labels in each tournament at each level. For $k \in \{2^{l-1} + 1, 2^l\}$, we used a pessimistic value of $k = 2^{l-1} + 1$ in the above formula to compute the bound, and compared it to the output of the program for $k = 2^l$. ■

Lemma 10. (Second Phase Depth Bound) *In any m -elimination tournament, the second phase has importance depth at most $\lceil m \rceil_2 - 1$ rounds for $m > 1$.*

Proof: When two labels are compared in round $i \geq 1$, the importance weight of their comparison is at most 2^{i-1} . Thus we have $\sum_{i=1}^{\lceil \log_2 m \rceil - 1} 2^{i-1} + \lceil m \rceil_2 = \lceil m \rceil_2 - 1$. ■

Putting everything together gives the importance depth theorem.

Theorem 11. (Importance Depth Bound) *For all m -elimination tournaments, the importance depth is upper bounded by*

$$\begin{cases} \lceil \log_2 k \rceil + m \lceil \log_2 (\lceil \log_2 k \rceil + 1) \rceil + \lceil m \rceil_2 \\ 1.5 \lceil \log_2 k \rceil + 3m + \lceil m \rceil_2 \\ \lceil \frac{k}{2} \rceil + 2m + \lceil m \rceil_2 \\ \text{For } k \leq 2^{62} \text{ and } m \leq 4\log_2 k, 2m + \lceil m \rceil_2 + 2\ln k + 2\sqrt{m \ln k}. \end{cases}$$

Proof: We simply add the depths of the first and second phases from Lemmas 9 and 10. For the last case, we bound $\sqrt{\ln k + 4(m-1)} \leq \sqrt{\ln k} + 2\sqrt{m}$ and eliminate subtractions in Lemma 10. ■

7. Lower Bound

All of our lower bounds hold for a somewhat more powerful adversary which is more natural in a game playing tournament setting. In particular, we disallow reductions which use importance weighting on examples, or equivalently, all importance weights are set to 1. Note that we can modify our upper bound to obey this constraint by transforming final elimination comparisons with importance weight i into $2i - 1$ repeated comparisons and use the majority vote. This modified construction has an importance depth which is at most m larger implying the ratio of the adversary and the reduction's regret increases by at most 1.

The first lower bound says that for any reduction algorithm B , there exists an adversary A with the average per-round regret r such that A can make B incur regret $2r$ even if B knows r in advance. Thus an adversary who corrupts half of all outcomes can force a maximally bad outcome. In the bounds below, f_B denotes the multiclass classifier induced by a reduction B using a binary classifier f .

Theorem 12. *For any deterministic reduction B from $k > 2$ classification to binary classification, there exists a choice of D and f such that $\text{reg}(f_B, D) \geq 2 \text{reg}(f, B(D))$.*

Proof: The adversary A picks any two labels i and j . All comparisons involving i but not j , are decided in favor of i . Similarly for j . The outcome of comparing i and j is determined by the parity of the number of comparisons between i and j in some fixed serialization of the algorithm. If the parity is odd, i wins; otherwise, j wins. The outcomes of all other comparisons are picked arbitrarily.

Suppose that the algorithm halts after some number of queries c between i and j . If neither i nor j wins, the adversary can simply assign probability $1/2$ to i and j . The adversary pays nothing while the algorithm suffers loss 1, yielding a regret ratio of ∞ .

Assume without loss of generality that i wins. The depth of the tournament is either c or at least $c+1$, because each label can appear at most once in any round. If the depth is c , then since $k > 2$, some label is not involved in any query, and the adversary can set the probability of that label to 1 resulting in $\rho(B) = \infty$.

Otherwise, A can set the probability of label j to be 1 while all others have probability 0. The total regret of A is at most $\lfloor \frac{c+1}{2} \rfloor$, while the regret of the winning label is 1. Multiplying by the depth bound $c + 1$, gives a regret ratio of at least 2. ■

Note that the number of rounds in the above bound can depend on A . Next, we show that for any algorithm B taking the same number of rounds for any adversary, there exists an adversary A with a regret of roughly one third, such that A can make B incur the maximal loss, even if B knows the power of the adversary.

Lemma 13. *For any deterministic reduction B to binary classification with number of rounds independent of the query outcomes, there exists a choice of D and f such that $\text{reg}(f_B, D) \geq (3 - \frac{2}{k}) \text{reg}(f, B(D))$.*

Proof: Let B take q rounds to determine the winner, for any set of query outcomes. We will design an adversary A with incurs regret $r = \frac{qk}{3k-2}$, such that A can make B incur the maximal loss of 1, even if B knows r .

The adversary's query answering strategy is to answer consistently with label 1 winning for the first $\frac{2(k-1)}{k}r$ rounds, breaking ties arbitrarily. The total number of queries that B can ask during this stage is at most $(k-1)r$ since each label can play at most once in every round, and each query occupies two labels. Thus the total amount of regret at this point is at most $(k-1)r$, and there must exist a label i other than label k with at most r losses. In the remaining $q - \frac{2(k-1)}{k}r = r$ rounds, A answers consistently with label i and all other skills being 0.

Now if B selects label 1, A can set $D(i | x) = 1$ with r/q average regret from the first stage. If B selects label i instead, A can choose that $D(1 | x) = 1$. Since the number of queries between labels i and k in the second stage is at most r , the adversary can incur average regret at most r/q . If B chooses any other label to be the winner, the regret ratio is unbounded. ■

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Appendix A. Table with Experimental Results

	J48				Logistic Regression			
Dataset(k)	Tree	FT	AP	APFT	Tree	FT	AP	APFT
arrhythmia (13)	37.64	36.37	34.32	34.97	55.27	55.04	40.44	34.97
audiology (24)	32.37	31.93	28.08	28.21	31.83	27.69	24.98	25.90
ecoli (8)	21.00	18.75	18.90	18.75	18.00	18.10	15.20	15.06
flare (7)	16.42	16.38	16.38	15.57	16.17	16.07	16.09	16.03
glass (6)	33.84	34.02	32.18	31.86	39.37	38.46	38.43	38.13
isolet (26)	27.30	24.60	12.40	14.60	35.30	26.50	8.40	8.40
kropt (18)	40.32	39.66	36.50	35.81	58.55	58.09	56.34	57.06
letter (25)	16.53	15.96	9.58	11.77	51.84	49.89	16.66	17.62
lymph (4)	25.22	22.28	21.83	22.28	24.32	24.20	23.86	24.07
nursery (5)	3.55	3.49	3.49	3.49	7.36	7.41	7.39	7.39
optdigits (10)	15.50	13.50	10.60	12.20	18.40	11.70	5.00	5.90
page-blocks (5)	2.99	2.84	3.00	2.95	4.06	3.31	3.12	3.21
pendigits (10)	8.00	7.60	7.00	7.60	23.40	22.40	6.10	5.10
satimage (6)	14.60	15.10	14.30	14.30	25.80	24.50	15.20	15.10
soybean (19)	15.70	13.00	13.00	13.00	16.80	16.50	13.60	13.60
vehicle (4)	30.86	31.11	31.57	28.93	21.60	21.37	20.78	20.31
vowel (11)	29.06	28.92	24.64	24.57	35.85	30.53	11.85	12.90
yeast (10)	44.04	44.21	43.99	44.06	45.13	43.66	42.28	43.26

Table A.1: Test error rates (in %) using J48 and logistic regression as binary learners. AP and FT stand for All-Pairs and Filter Tree respectively. APFT is the All-Pairs variant of the Filter Tree.